Noncommutativity and Model Building

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Abstract: We propose a way to introduce matter fields transforming in arbitrary representations of the gauge group in noncommutative $U(N)$ gauge theories. We then argue that in the presence of supersymmetry, an ordinary commutative $SU(N)$ gauge theory with a general matter content can always be embedded into a noncommutative $U(N)$ theory at energies above the noncommutativity mass scale $M_{NC} \sim \theta^{-1/2}$. At energies below M_{NC}, the $U(1)$ degrees of freedom decouple due to the IR/UV mixing, and the noncommutative theory reduces to its commutative counterpart. Supersymmetry can be spontaneously broken by a Fayet-Iliopoulos D-term introduced in the noncommutative $U(N)$ theory. $U(1)$ degrees of freedom become arbitrarily weakly coupled in the infrared and naturally play the rôle of the hidden sector for supersymmetry breaking. To illustrate these ideas we construct a noncommutative $U(5)$ GUT model with Fayet-Iliopoulos supersymmetry breaking, which reduces to a realistic commutative theory in the infrared.

Keywords: Non-Commutative Geometry, Supersymmetry Breaking, Gauge Symmetry.

The attempts to construct realistic models of particle physics in the framework of noncommutative gauge field theories encounters two outstanding problems. First, the mixing between the infrared (IR) and the ultraviolet (UV) degrees of freedom discovered in [1] leads to novel singularities in the low-energy effective action, further analysed in [2–9] (for recent reviews and a more extensive list of references see [10, 11]). Second, there is an apparent obstruction [12] in constructing representations of matter fields other than (anti)-fundamental and adjoint. In this letter we show how to construct matter fields transforming in arbitrary representations of the noncommutative $U(N)$ gauge group. We also explain that in supersymmetric theories the IR/UV mixing does not lead to problems, and moreover becomes an important ingredient of realistic model building.

1. It is well-known that in non-supersymmetric theories the IR/UV mixing leads to IR poles in the effective action. This leads to drastic modifications of the dispersion relations for the photon [2] and to large Lorentz-violating effects [13] and also threatens the renormalizability of the theory when these poles are included at the higher loop level. Importantly, however, this generic picture does not apply to supersymmetric theories, where the IR/UV mixing is at most logarithmic, and does not lead to either unconventional dispersion relations or large Lorentz-violating effects. Moreover this milder form of IR/UV mixing leads to the decoupling of the $U(1)$ degrees of freedom in the IR. Not only the $U(1)$ degrees of freedom become free in the IR [6], they also trigger spontaneous supersymmetry breaking [7] in the presence of an appropriate Fayet-Iliopoulos D-term and play the rôle of the hidden sector.

The leading order terms in the derivative expansion of the Wilsonian effective action for supersymmetric noncommutative QCD were analysed in [7]. To illustrate the decoupling of the $U(1)$ sector, it is sufficient to look at the supersymmetric pure $U(N)$ gauge theory:

$$\mathcal{L}_{\rm eff} \;=\; -14g_{1}^2(k)\; F_{\mu\nu}^{U(1)}F_{\mu\nu}^{U(1)} \;-\; 14g_{N}^2(k)\; F_{\mu\nu}^{SU(N)}F_{\mu\nu}^{SU(N)} \;+\; \cdots \;, \tag{1}$$

where the dots stand for terms involving fermions and higher-derivative corrections. The multiplicative coefficients in front of the gauge kinetic terms in (1) define the Wilsonian coupling constants. The running of the $U(1)$ has the following asymptotic behaviour:

$$1g_{1}^2(k) \to \pm 3N(4\pi)^2 \log k^2 \;, \tag{2}$$

where the plus (minus) sign corresponds to $k^2 \to \infty$ ($k^2 \to 0$), whereas for the $SU(N)$ gauge factor we have in both limits:

$$1g_{N}^2(k) \to 3N(4\pi)^2 \log k^2 \;. \tag{3}$$

The change in the running of the $U(1)$ coupling in (2) occurs at the scale $k^2 \sim M_{NC}^2$, where $M_{NC} \sim \theta^{-1/2}$ is the noncommutative mass and θ the usual noncommutativity parameter. This running was interpreted in [7] as having a full noncommutative $U(N)$ gauge theory in the UV, which in the low-energy limit appears as a commutative $SU(N)$ theory with a decoupled free $U(1)$ factor. Note that this decoupling in the IR does not mean that the noncommutative $U(N)$ gauge symmetry is broken. In fact, there is a gauge-invariant completion of (1) proposed in [8,9] which involves open Wilson lines [14]. This completion of (1) introduces higher-derivative terms which are irrelevant for the low-energy dynamics.

Finally, note that supersymmetry can be spontaneously broken by introducing a Fayet-Iliopoulos D-term in the lagrangian

$$\mathcal{L}_{FI} = \xi_{FI} \int d^2\theta d^2\bar{\theta}\ \mathrm{tr}_N V \ , \tag{4}$$

where V is the real $U(N)$ vector superfield and the trace over the N by N matrices selects the $U(1)$-component of V. The Fayet-Iliopoulos action, $\int d^4x L_{FI}$, is $U(N)$ gauge invariant and can be naturally introduced at tree-level in any $U(N)$ theory. This provides us with a scenario of a gauge-mediated supersymmetry breaking where the $U(1)$ degrees of freedom, which eventually become arbitrarily weakly coupled in the IR, play the rôle of the hidden sector [7]. Both the hidden sector and the messenger sector are naturally part of the noncommutative $U(N)$ gauge theory.

2. Now we discuss how to introduce matter fields transforming in general representations of noncommutative $U(N)$. The construction of fundamental, $f^i(x)$, anti-fundamental, $\tilde{f}_i(x)$, and adjoint, $\phi^i_j(x)$ representations of $U(N)$ is straightforward,

$$f^i \to U^i_{i'} * f^{i'} \ , \quad \tilde{f}_i \to \tilde{f}_{i'} * (U^{-1})^{i'}_i \ , \quad \phi^i_j \to U^i_{i'} * \phi^{i'}_{j'} * (U^{-1})^{j'}_j \ , \tag{5}$$

where the $*$-product is the usual Weyl-Moyal deformation, $(f * g)(x) \equiv f(x) e^{i2\theta^{\mu\nu}\overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu} g(x)$, and $U \in U(N)$. Consider now other representations, for example, a rank two representation $t^{ij}(x)$. The naive noncommutative gauge transformation, $t^{ij} \longrightarrow U^i_{i'} * U^j_{j'} * t^{i'j'}$, is not correct as the closure property of the group multiplication,

$$(t^U)^V = t^{U*V} \ , \tag{6}$$

is broken due to the noncommutativity of the $*$-product. It is easy to convince oneself that rearranging the positions of the U's or any other straightforward modification of the naive transformation does not help. This is in fact a generic problem well-known in noncommutative geometry.

To resolve this problem we propose to modify the transformation laws for the matter fields in a non-trivial, gauge field-dependent way. To this end we first introduce gauge-singlet matter fields, $\mathcal{T}^{ij}(x)$, as in the construction of noncommutative baryons of Ref. [15], making use of the following open Wilson line

$$W(x) = P_* \exp\left(i \int_0^1 d\sigma\ d\zeta^\mu d\sigma A_\mu(x + \zeta(\sigma)) \right) \ , \tag{7}$$

where the integration is along the contour C_∞ from ∞ to x,

$$C_\infty = \{\zeta_\mu(\sigma),\ 0 \leq \sigma \leq 1 \mid \zeta(0) = \infty,\ \zeta(1) = 0\}\ , \tag{8}$$

and the path ordering is with respect to the star product. The shape of the contour is not important for our construction, but for concreteness one can always consider straight rays such as the one from (y_0, y_1, y_2, ∞) to (x_0, x_1, x_2, x_3). Under a noncommutative gauge transformation, $A_\mu \to U * (A_\mu - i\partial_\mu) * U^{-1}$, the Wilson line (7) transforms as

$$W(x) \to U_\infty * W(x) * U^{-1}(x)\ . \tag{9}$$

The key element of our construction is to restrict the allowed gauge transformations $U(x)$ to those which, as $x \to \infty$ *along the contour* (8), approach a constant U_∞, with vanishing derivatives to all orders. Then, without loss of generality we can set $U_\infty = 1$, and as a result (9) becomes

$$W(x) \to W(x) * U^{-1}(x)\ . \tag{10}$$

This restriction can be motivated in a number of ways. For example, it is compulsory when the theory is compactified on a 4-sphere. Also the Noether charges associated to the gauge symmetry transform covariantly (form an algebra) only under gauge transformations which approach the identity at spatial infinity [16]. In addition, if we first fix the $A_0 = 0$ gauge, the requirement of $U(|\vec{x}| = \infty) = 1$ is necessary to project onto the states which satisfy the Gauss' law [17].

We can now associate to t^{ij} a gauge-singlet field $\mathcal{T}^{ij}$ defined by

$$\mathcal{T}^{ij} = W^i_{i'} * W^j_{j'} * t^{i'j'}\ . \tag{11}$$

The invariance of $\mathcal{T}^{ij}$ determines the transformations of t^{ij} under the noncommutative gauge group,

$$(t^U)^{ij} = (U * W^{-1})^j_k * U^i_l * W^k_m * t^{lm}\ . \tag{12}$$

In tensor notation, it reads

$$t^U = (1 \otimes U * W^{-1}) * (U \otimes W) * t\ . \tag{13}$$

Remarkably, this transformation satisfies the closure property of group multiplication (6). In the commutative limit the Wilson lines in (13) will cancel each other and the transformation law will reduce to the conventional one.

The same construction applies to all higher-rank representations. For a generic rank-n representation $t_{[n]}$, the generalization of (13) reads

$$t^U_{[n]} = (U * W^{-1})^{\epsilon_n}_n * (U * W^{-1})^{\epsilon_{n-1}}_{n-1} * \cdots * (U * W^{-1})^{\epsilon_1}_1 * (W^{\epsilon_1} \otimes \cdots \otimes W^{\epsilon_n}) * t_{[n]}\ . \tag{14}$$

Here we have defined

$$(U * W^{-1})_i^{\epsilon_i} \equiv 1 \otimes \cdots \otimes (U * W^{-1})^{\epsilon_i} \otimes \cdots \otimes 1 \ , \tag{15}$$

with $(U * W^{-1})^{\epsilon_i}$ in the i^{th} position in the tensor product, and the power $\epsilon_i = +1\,(-1)$ if i is an upper (lower) index. Equation (14) follows from the invariance of the rank-n gauge-singlet field

$$\mathcal{T}_{[n]} = (W^{\epsilon_1} \otimes \cdots \otimes W^{\epsilon_n}) * t_{[n]} \ . \tag{16}$$

Irreducible representations are obtained from the reducible ones in the same way as in the commutative case.

Remarkably, the gauge-singlet matter fields $\mathcal{T}(x)$ introduced above are related to $t(x)$ by a gauge transformation $U = W$,

$$\mathcal{T} = t^W \ . \tag{17}$$

By the same token we now introduce the gauge-singlet vector field

$$\mathcal{A}_\mu \equiv A_\mu^W = W * (A_\mu - i\partial_\mu) * W^{-1} \ , \tag{18}$$

and write down the appropriate gauge-singlet 'covariant' derivative for matter fields. In the rank 2 case we have

$$\mathcal{D}_\mu = (1 \otimes 1)\partial_\mu + i(\mathcal{A}_\mu \otimes 1) + i(1 \otimes \mathcal{A}_\mu) \ , \tag{19}$$

and the generalization to the rank-n case is obvious. With these ingredients we can now construct a gauge-invariant action for the matter field t

$$\int d^4x \,\mathrm{tr}\, |\mathcal{D}_\mu * t^W|^2 = \int d^4x \,\mathrm{tr}\, |D_\mu * t|^2 \ , \qquad \text{for scalars,} \tag{20}$$

$$i \int d^4x \,\mathrm{tr}\, \bar{t}^W * \gamma^\mu \mathcal{D}_\mu * t^W = i \int d^4x \,\mathrm{tr}\, \bar{t} * \gamma^\mu D_\mu * t \ , \quad \text{for fermions,} \tag{21}$$

where D_μ is defined such that $\mathcal{D}_\mu * t^W = (D_\mu * t)^W$. It then follows that

$$D_\mu = (W \otimes W)^{-1} * \big((1 \otimes 1)\partial_\mu + i(A_\mu^W \otimes 1) + i(1 \otimes A_\mu^W)\big) * (W \otimes W) \ , \tag{22}$$

and A_μ^W was defined in (18). The action written in terms of the original variables $t(x)$ and $A_\mu(x)$ is gauge-invariant, but takes a cumbersome form which is not tractable in perturbation theory: Taylor-expanding the Wilson lines would lead to non-renormalizable vertices and is not a good idea as it misses the fact that $W(x)$ is a $U(N)$ group element. For practical applications, one first has to perform a gauge transformation using $U(x) = W(x)$ on the original variables $t(x)$ and $A_\mu(x)$, arriving at the gauge-singlet variables as in (17), (18). The action takes now exactly the same form as in the commutative theory (with star products). This transition to the gauge-singlet variables is nothing but a gauge-fixing procedure. For example, we can choose $W(x)$ to be a straight Wilson line parallel to the x_3 axis. This Wilson line is precisely the gauge transformation used to fix the $A_3 = 0$ gauge. Moreover, the usual residual x_3-independent gauge transformations are not allowed since $U_\infty = 1$, the gauge fixing is complete and no further gauge transformations are possible. Hence we can interpret the gauge-singlet fields as the degrees of freedom of the completely gauge-fixed formulation. Wilson lines and non-renormalizable interactions are absent in this 'physical' gauge which is suitable for perturbative calculations.

A few remarks are in order. First, in the ordinary commutative case one can still carry out this construction, which however trivially reduces to the usual approach with standard gauge transformations, since the Wilson lines would cancel in (13) and (14) as remarked earlier. Second, the same comment applies also to the noncommutative theory with (anti)-fundamental and adjoint matter fields. Third, the supersymmetrization of this construction is immediate in terms of component fields. To establish a gauge-invariant formulation in the superfield formalism, one can similarly introduce gauge-singlet vector superfields $\mathcal{V}(x,\vartheta,\bar\vartheta)$ (at least in the Wess-Zumino gauge)

$$e^{2\mathcal{V}} = (W^{-1})^\dagger(\bar y) * e^{2V(x,\vartheta,\bar\vartheta)} * W^{-1}(y) \;, \tag{23}$$

where y is the usual chiral coordinate in the superspace and the singlet chiral matter superfields $\mathcal{T}(y,\vartheta)$ in the rank 2 case are given by

$$\mathcal{T}(y,\vartheta) = (W(y)\otimes W(y)) * t(y,\vartheta) \;. \tag{24}$$

Then the action $S[\mathcal{V},\mathcal{T}]$ takes the standard form $\int d^4x \int d^2\vartheta d^2\bar\vartheta\, \mathcal{T}^\dagger * (e^{2\mathcal{V}}\otimes e^{2\mathcal{V}}) * \mathcal{T}$. Finally, we note that we can also couple $\mathrm{tr}\, e^{2\mathcal{V}}$ to the matter fields $\mathcal{T}$ in the theory. This interaction has no analogue in the commutative $SU(N)$ theory.

In the presence of supersymmetry, the IR/UV mixing amounts to nothing more than the decoupling in the low-energy limit of the overall $U(1)$ factor, as explained in **1.** Using our construction of general representations, an ordinary commutative $SU(N)$ gauge theory with a general matter content can now always be embedded into a supersymmetric noncommutative $U(N)$ theory at energies above the noncommutativity mass scale M_{NC}.

3. To put at work the machinery we have discussed, we now construct a simple supersymmetric noncommutative unified theory (NUT) with gauge group $U(5)$, with an adjoint Higgs superfield Φ and three generations of matter superfields $\tilde{\mathcal{F}}$ and $\mathcal{T}$ in the anti-fundamental and in the rank 2 antisymmetric representations generalizing [18]. We work in the physical gauge introduced in **2**. The matter fields are coupled to the gauge fields as follows:

$$\int d^4x \int d^2\vartheta d^2\bar\vartheta \left(\tilde{\mathcal{F}} * K_{\tilde{\mathcal{F}}} * \tilde{\mathcal{F}}^\dagger \;+\; \mathcal{T}^\dagger * (K_{\mathcal{T}}\otimes K_{\mathcal{T}}) * \mathcal{T}\right) \;, \tag{25}$$

where the gauge kernels are

$$K_{\tilde{\mathcal{F}}} = e^{-2\mathcal{V}} + \mathbb{1}\,\kappa_{\tilde{\mathcal{F}}} N(N - \mathrm{tr}\, e^{-2\mathcal{V}}) \ , \quad K_{\mathcal{T}} = e^{2\mathcal{V}} + \mathbb{1}\,\kappa_{\mathcal{T}} N(N - \mathrm{tr}\, e^{2\mathcal{V}}) \ , \tag{26}$$

and $\kappa_{\tilde{\mathcal{F}}}$ and $\kappa_{\mathcal{T}}$ are arbitrary constants. Furthermore, to break supersymmetry we introduce the Fayet-Iliopoulos term (4).

Below the noncommutativity mass scale the IR/UV mixing triggers the decoupling of the overall $U(1)$ degrees of freedom from the $SU(5)$ as explained in **1.** Then the $SU(5)$ gauge symmetry gets broken by the Higgs vacuum expectation value

$$\langle \phi \rangle = M_{NUT}\, \mathrm{diag}(1, 1, 1, -3/2, -3/2) \tag{27}$$

below the NUT scale $M_{NUT} < M_{NC}$ to $SU(3) \times SU(2) \times U(1)$. The matter fields decompose into representations of $SU(3) \times SU(2)$ as $\bar{\mathbf{5}} = (\bar{\mathbf{3}}, \mathbf{1}) + (\mathbf{1}, \mathbf{2})$ and $\mathbf{10} = (\mathbf{3}, \mathbf{2}) + (\bar{\mathbf{3}}, \mathbf{1}) + (\mathbf{1}, \mathbf{1})$.

We can now solve the D-flatness conditions:

$$D^0 = \sqrt{2/N}(1 - \kappa_{\mathcal{T}})\, \mathcal{T}^\dagger \mathcal{T} - \sqrt{2/N}(1 - \kappa_{\tilde{\mathcal{F}}})\, \tilde{\mathcal{F}} \tilde{\mathcal{F}}^\dagger - \xi_{FI}, \tag{28}$$

$$D^a = 2\mathcal{T}^\dagger (1 \otimes T^a) \mathcal{T} - 2\tilde{\mathcal{F}} T^a \tilde{\mathcal{F}}^\dagger \ , \quad a = 1, \ldots, N^2 - 1 \ , \ (N = 5) \ . \tag{29}$$

In general, $\langle D^a \rangle = 0$ and $\langle D^0 \rangle = -\xi_{FI}$ and supersymmetry is broken spontaneously for $\xi_{FI} \neq 0$. For the choice $\kappa_{\tilde{\mathcal{F}}} = 0$ and $\kappa_{\mathcal{T}} = 2$ we get a phenomenologically interesting supersymmetry breaking mass spectrum, in which the masses for all the matter scalar fields (squarks and sleptons) are shifted by a positive amount $\sim \sqrt{\xi_{FI}}$.

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